Cauchy-Schwarz Problem Sheet

Adithya Bhaskar

February 2016

1 Introduction

Hi! Here is a problem sheet on Cauchy-Schwarz. Happy problem solving!

2 Problems dude! :D

1. (Czech-Slovak 1999) Let $a, b, c > 0$. Prove that

$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b} \geq 1.$$

2. (Romania 1997) For positive reals a, b, c show that

$$\sum_{cyc} \frac{a^2}{a^2+2bc} \geq 1 \geq \sum_{cyc} \frac{bc}{a^2+2bc}.$$

3. (Hungary 1996, trivial) If $a + b = 1, a, b > 0$ then prove that

$$\frac{a}{a+1} + \frac{b}{b+1} \geq \frac{1}{3}.$$

4. (Romania TST) Let a, b, x, y, z be positive reals. Prove the inequality

$$\frac{x}{ay+bz} + \frac{y}{az+bx} + \frac{z}{ax+by} \geq \frac{3}{a+b}.$$

5. (Turkey) Let $n \geq 2$ be a positive integer, and $x_1, ..., x_n \in R$ such that $x_1^2 + ... + x_n^2 = 1$. Determine the smallest possible value of

$$\frac{x_1^5}{x_2+...x_n} + \frac{x_2^5}{x_3+...+x_n+x_1} + ... + \frac{x_n^5}{x_1+...+x_{n-1}}.$$

6. (Darij Grinberg) Show that for all positive reals a, b, c,

$$\frac{\sqrt{b+c}}{a} + \frac{\sqrt{c+a}}{b} + \frac{\sqrt{a+b}}{c} \geq \frac{4(a+b+c)}{\sqrt{(a+b)(b+c)(c+a)}}.$$

7. (Cezar Lupu) Let a, b, c be positive reals such that $a+b+c+abc = 4$. Prove that

$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{c+a}} + \frac{c}{\sqrt{a+b}} \geq \frac{a+b+c}{\sqrt{2}}.$$

8. (Titu Andreescu) Show that for all nonzero reals a, b, c one has

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\ge\frac{a}{c}+\frac{c}{b}+\frac{b}{a}.$$

9. (IMO 1992) Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the orthogonal projections of S onto the yz, zx, xy planes, respectively. Show that

$$|S|^2 \le |S_x||S_y||S_z|.$$

10. (Ivan Matic) Let $a_1, ..., a_n$ be positive real numbers. Then show that

$$\frac{a_1^3}{a_2}+...+\frac{a_n^3}{a_1}\ge a_1^2+...+a_n^2.$$

11. (Po-Shen Loh) Let a, b, c be positive reals such that $abc = 1$. Prove that

$$\sum_{cyc}\frac{1}{a+b+1}\le 1.$$

12. Let $a, b, c > 0$ such that $a+b+c = a^3+b^3+c^3$.Prove that:

$$\frac{a}{a^2+1}\left(\frac{b}{c}\right)^2+\frac{b}{b^2+1}\left(\frac{c}{a}\right)^2+\frac{c}{c^2+1}\left(\frac{a}{b}\right)^2\ge\frac{a+b+c}{2}.$$

13. (Problem of the Month, Greece; easy) If x, y, z are positive reals, prove that

$$\frac{(y+z)^2}{y+z+2x}+\frac{(z+x)^2}{z+x+2y}+\frac{(x+y)^2}{x+y+2z}\ge\frac{(\sqrt{x}+\sqrt{y}+\sqrt{z})^2}{3}.$$

14. (JBKMO) Show that

$$\frac{1+a^2}{1+b+c^2}+\frac{1+b^2}{1+c+a^2}+\frac{1+c^2}{1+a+b^2}\ge 2.$$

for reals $a, b, c \ge -1$.

15. (IMO 1995) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\ge\frac{3}{2}.$$